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BOUNDEDNESS IN DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. We study boundedness of solutions of dynamic equations on time scales by using Lyapunov-type functions.

1. Introduction

The qualitative theory for differential equations was begun by H. Poincaré(1881) and A. M. Lyapunov (1892). It is well known that Lyapunov direct method plays the key role in the stability study of dynamical systems. Historically, Lyapunov presented four celebrated original theorems on stability, asymptotic stability and instability, which are now called the principal theorems of stability which are fundamental to stability of dynamical systems.

In this paper we study the boundedness of the zero solution of the first order vector dynamic equation

$$\begin{cases} x^{\Delta} = f(t,x), \ t \in \mathbb{T}, \\ x(t_0) = x_0, \ t_0 \in \mathbb{T}, \ x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $t \geqslant t_0$ and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous function.

Recently Peterson and Tisdell [11] used Lyapunov-type functions to prove that the solution of Eqs.(1.1) is uniformly bounded. Also, Peterson and Raffoul [10] investigated exponential stability of the solution of Eqs.(1.1). Choi et al.[4] studied the h-stability for linear dynamic systems on time scales.

If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = x'$ and (1.1) is the following initial value problem for ordinary differential equations

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0. \end{cases} \tag{1.2}$$

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If $\mathbb{T} = \mathbb{Z}$, then $x^\Delta = \Delta x$, and (1.1) is the following initial value problem for ordinary difference equations

$$\begin{cases} x(n+1) - x(n) = f(n, x(n)), \ n \geq 0, \\ x(n_0) = x_0, \ n_0 \geq 0. \end{cases} \tag{1.3}$$

2. Preliminaries

In 1988 Stefan Hilger introduced the concept of the time scale calculus in his Ph.D. dissertation [9] as a mean to unify continuous and discrete analysis. Many results one encounters in the study of both differential and difference equations have analogs in the time scale case. However, the time scale result encompasses both the discrete and continuous results as special cases.

The following definitions and theorems can be found in books by Bohner and Peterson [1, 2] and DaCunha's paper [7].

Definition 2.1. A *time scale* $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$.

Definition 2.2. The *forward jump operator* $\sigma(t)$, and the *backward jump operator* $\rho(t)$, are defined by

$$\sigma(t) = inf\{s \in \mathbb{T}, s > t\}, \qquad \rho(t) = sup\{s \in \mathbb{T}, s < t\},$$

respectively.

Definition 2.3. An element $t \in \mathbb{T}$ is *left-dense*, *right-dense*, *left-scattered*, *right-scattered*, if

$$\rho(t) = t, \quad \sigma(t) = t, \quad \rho(t) < t, \quad \sigma(t) > t,$$

respectively.

Definition 2.4. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called *graininess function*.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.
$\mathbb{T}^k = \mathbb{T} - \{m\}$ when $\mathbb{T}$ has a left-scattered maximum m, and $\mathbb{T}^k = \mathbb{T}$ otherwise.

Definition 2.5. Let $f : \mathbb{T} \to \mathbb{R}^n$, f is called *differentiable* at $t \in \mathbb{T}^k$ with (delta) derivative $f^\Delta(t) \in \mathbb{R}^n$, if given $\epsilon > 0$ there exists a neighborhood U of t such that for all $s \in U$

$$|f_i^\sigma(t) - f_i(s) - f_i^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|,$$

where $f_i : \mathbb{T} \to \mathbb{R}, \ i = 1, 2, \cdots, n$.

If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = f(t+1) - f(t)$.

THEOREM 2.6. *Suppose $f : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}^k$.*

(i) *If f is delta differentiable at t, then f is continuous at t;*

(ii) *If f is continuous at t and t is right-scattered, then f is delta differentiable at t and*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If f is delta differentiable at t and t is right-dense, then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is delta differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Again we consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, Theorem 2.6 (iii) yields that $f : \mathbb{R} \to \mathbb{R}^n$ is delta differentiable at $t \in \mathbb{R}$ iff

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists, i.e., iff f is differentiable (in the ordinary sense) at t.

In this case we then have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

(ii) If $\mathbb{T} = \mathbb{Z}$, Theorem 2.6 (ii) yields that $f : \mathbb{Z} \to \mathbb{R}^n$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t),$$

where Δ is the usual forward difference operator defined by the last equation above.

DEFINITION 2.7. The function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be *right-dense continuous* if

(i) f is continuous at every right-dense point $t \in \mathbb{T}$, and

(ii) $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ will be denoted in this paper by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

DEFINITION 2.8. The mapping $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if

$$1 + \mu(t)p(t) \neq 0, \qquad t \in \mathbb{T}^k,$$

The set of all regressive and rd-continuous functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$.

For $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, we define the *exponential function* by

$$e_p(t, s) = exp[\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau], \qquad s, t \in \mathbb{T},$$

where the cylinder transformation $\xi_{\mu(\tau)}(p(\tau)) = \frac{1}{\mu(\tau)} Log(1 + \mu(\tau)p(\tau))$.

THEOREM 2.9. *If $p, q \in \mathcal{R}$, then we have, for all $t, s, r \in \mathbb{T}$*

(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ *and* $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$ *and* $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$.

THEOREM 2.10. *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$. Then:*

(i)The sum $f + g$ is differentiable at t with

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) For any constant α, αf is differentiable at t with

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ are differentiable at t with

$$\begin{aligned}(fg)^\Delta(t) &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)\\ &= f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).\end{aligned}$$

DEFINITION 2.11. Let $f \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$., Then $g : \mathbb{T} \to \mathbb{R}^n$ is called *antiderivative* of f on $\mathbb{T}$, if it is differentiable on $\mathbb{T}$ and satisfies $g^\Delta = f(t)$ for $t \in \mathbb{T}$. In this case, we define

$$\int_a^t f(s)\Delta s = g(t) - g(a), \qquad a \leqslant t \quad \text{and} \quad t \in \mathbb{T}.$$

THEOREM 2.12. *(Chain Rule). Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \{\int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh\}g^\Delta(t)$$

holds.

Let $\mathbb{T}_0 := \{t \in \mathbb{T} : t \geq t_0\}$. Assume $V : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^+$ is delta differentiable in variable t and continuously differentiable in variable x, and $x(t)$ is any solution of dynamic system (1.1)., Then from [6, 8] we know the delta derivative along $x(t)$ for $V(t,x)$ is the following

$$\begin{aligned} V^\Delta(t,x) &= V^\Delta(t,x(t)) \\ &= V_t^\Delta(t,x(\sigma(t))) + \int_0^1 V_x'(t,x(t)+h\mu(t)x^\Delta(t))dhx^\Delta(t) \\ &= V_t^\Delta(t,x(\sigma(t))) + \int_0^1 V_x'(t,x(t)+h\mu(t)x^\Delta(t))dhf(t,x), \end{aligned}$$

where V_t^Δ is considered as the delta derivative in the first variable t and V_x' is taken as the normal derivative in variable x. Then we call $V(t,x)$ a *Lyapunov-type function* on time scales.

3. Boundedness in dynamic equations on time scales.

In this section we investigate the boundedness of solution to first-order dynamic equations on time scales.

Definition 3.1. We say that a solution of (1.1) is *bounded* if there exists a constant $C(t_0,x_0)$ such that

$$\|x\| \leq C(t_0,x_0) \text{ for } t \in \mathbb{T}_0.$$

where C is a constant and depends on t_0. Moreover, solutions of (1.1) are *uniformly bounded* if C is independent of t_0.

Theorem 3.2. [11] *Assume that $D \subset \mathbb{R}^n$ and there exists a Lyapunov function $V : \mathbb{T}_0 \times D \to \mathbb{R}_+$ such that for all $(t,x) \in \mathbb{T}_0 \times D$:*

$$\begin{aligned} &V(t,x) \to \infty \text{ as } \|x\| \to \infty; \\ &V(t,x) \leq \lambda_2 \|x\|^q; \\ &V^\Delta(t,x) \leq \frac{-\lambda_3\|x\|^r + L}{1+M\mu(t)}; \\ &V(t,x) - V^{r/q}(x) \leq \gamma; \end{aligned}$$

where $\lambda_2, \lambda_3, q, r$ are positive constants; L and γ are nonnegative constants, and $M := \lambda_3/\lambda_2^{r/q}$. Then all solutions of Eq.(1.1) that stay in D are uniformly bounded.

THEOREM 3.3. [11] *Assume that* $D \subset \mathbb{R}^n$ *and there exist a Lyapunov function* $V : \mathbb{T}_0 \times D \to \mathbb{R}_+$ *such that for all* $(t, x) \in \mathbb{T}_0 \times D$:

$$V(t,x) \to \infty \text{ as } \|x\| \to \infty; \tag{3.1}$$

$$V^{\Delta}(t,x) \leq \frac{-\lambda_3 V(t,x) + L}{1 + \lambda_3 \mu(t)}; \tag{3.2}$$

where $\lambda_3 > 0$ *and* $L \geq 0$ *are constants. Then all solutions of Eq.(1.1) that stay in* D *are uniformly bounded.*

Next, we give more generalized boundedness theorems.

THEOREM 3.4. *Assume* $D \subset \mathbb{R}^n$ *contains the origin and there exist a Lyapunov-type function* $V : \mathbb{T}_0 \times D \to \mathbb{R}_+$ *and* $h : \mathbb{T} \to \mathbb{R}^+$ *that* $h(t)$ *satisfies bounded and* $\frac{h^{\Delta}(t)}{h(t)}$ *is regressive. Furthermore, suppose that*

$$a\|x\|^p \leq V(t,x), \tag{3.3}$$

$$V^{\Delta}(t,x) \leq \frac{-V(t,x) + L}{h(t) + \mu(t)h^{\Delta}(t)} h^{\Delta}(t), \tag{3.4}$$

where a *and* p *are positive constants,* L *is a nonnegative constant. Then all solutions of Eq.(1.1) that stay in* D *satisfy*

$$\|x\| \leq \Big(\frac{L + (V(t_0, x_0) + L)\frac{h(t_0)}{h(t)}}{a}\Big)^{1/p}. \tag{3.5}$$

Proof. Let x be a solution of (1.1) that stay in D for all $t \in \mathbb{T}_0$. Consider

$$\begin{aligned}
&[V(t,x)h(t)]^{\Delta} \\
&= V^{\Delta}(t,x)h(\sigma(t)) + V(t,x)h^{\Delta}(t) \\
&\leq \frac{-V(t,x) + L}{h(t) + \mu(t)h^{\Delta}(t)} h^{\Delta}(t)(h(t) + \mu(t)h^{\Delta}(t)) + V(t,x)h^{\Delta}(t) \\
&\leq Lh^{\Delta}(t)
\end{aligned}$$

Integrating both sides from t_0 to t with $x_0 = x(t_0)$, we obtain

$$V(t,x)h(t) \leq Lh(t) + (V(t_0, x_0) + L)h(t_0)$$

$$V(t,x) \leq L + (V(t_0, x_0) + L)\frac{h(t_0)}{h(t)}, \quad \text{by (3.3)}$$

$$\|x\| \leq \Big(\frac{L + (V(t_0, x_0) + L)\frac{h(t_0)}{h(t)}}{a}\Big)^{1/p}. \tag{3.6}$$

$\square$

COROLLARY 3.5. *Assume that the conditions of Theorem 3.4 hold. Let L=0, $1/h(t)$ bounded and*

$$V(t,x) \leq b\|x\|^p. \tag{3.7}$$

Then by (3.5) and (3.7) we have

$$\|x\| \leq (\frac{L + (b\|x_0\|^p + L)\frac{h(t_0)}{h(t)}}{a})^{1/p}$$

$$\|x\| \leq (\frac{(b\|x_0\|^p)\frac{h(t_0)}{h(t)}}{a})^{1/p}. \tag{3.8}$$

Hence we obtain

$$\|x\| \leq c\|x_0\| H(t) H(t_0)^{-1},\ t \geq t_0$$

where $H(t) = (1/h(t))^{1/p}$, $c = (\frac{b}{a})^{1/p}$.

Next we will give special case of Theorem 3.4.

If we set $\mathbb{T} = \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots n_0+k, \cdots\}$, then Theorem 3.4 also holds.

COROLLARY 3.6. *Assume $D \subset \mathbb{R}^n$ contains the origin and there exist a Lyapunov-type function $V : \mathbb{N}(n_0) \times D \to \mathbb{R}_+$ and $h : \mathbb{N}(n_0) \to \mathbb{R}^+$ that $h(n)$ satisfies bounded and $\frac{\Delta h(n)}{h(n)}$ is regressive. Furthermore, suppose that*

$$a\|x\|^p \leq V(n,x), \tag{3.9}$$

$$V^{\Delta}(n,x) \leq \frac{-V(n,x) + L}{h(n) + \Delta h(n)} \Delta h(n), \tag{3.10}$$

where a and p are positive constants, L is a nonnegative constant. Then all solutions of (1.3) that stay in D satisfy

$$\|x\| \leq (\frac{L + (V(n_0,x_0) + L)\frac{h(n_0)}{h(n)}}{a})^{1/p}. \tag{3.11}$$

COROLLARY 3.7. *Assume that the conditions of Corollary 3.6 hold. Let L=0, $1/h(n)$ bounded and*

$$V(n,x) \leq b\|x\|^p. \tag{3.12}$$

Then by (3.11) and (3.12) we have

$$\|x\| \leq \left(\frac{L + (b\|x_0\|^p + L)\frac{h(n_0)}{h(n)}}{a}\right)^{1/p}$$

$$\|x\| \leq \left(\frac{(b\|x_0\|^p)\frac{h(n_0)}{h(n)}}{a}\right)^{1/p}. \tag{3.13}$$

Hence we obtain

$$\|x\| \leq c\|x_0\|H(n)H(n_0)^{-1}, \; n \geq n_0,$$

where $H(n) = (1/h(n))^{1/p}$, $c = (\frac{b}{a})^{1/p}$.

4. Examples

EXAMPLE 4.1. *Consider the following system of IVP for* $t \geq t_0 \geq 0$

$$\begin{cases} x_1^\Delta = -ax_1 + ax_2, \\ x_2^\Delta = -ax_1 - ax_2, \\ (x_1(t_0), x_2(t_0)) = (c, d). \end{cases} \tag{4.1}$$

where $a > 0$, c *and* d *are certain constants. If there is a bounded function* $h : \mathbb{T} \to (0, \infty)$ *such that for all* $a \in \mathbb{R}^+$

$$\frac{h^\Delta(t)}{h(t) + h^\Delta(t)\mu(t)} \leq 2a(1 - a\mu(t)), \tag{4.2}$$

then all solutions to Eq.(4.1) satisfy (3.5).

Proof. We will show that, under the above assumptions, the conditions of Theorem 3.4 are satisfied. Choose $D = \mathbb{R}^2$ and $V(x) = x_1^2 + x_2^2$.

$$\begin{aligned} \dot{V}(t, x) &= 2x \cdot f(t, x) + \mu(t)\|f(t, x)\|^2 \\ &= -2a(1 - a\mu(t))\|x\|^2 \\ &\leq -\frac{h^\Delta(t)}{h(t) + h^\Delta(t)\mu(t)}\|x\|^2 \\ &= \frac{-V(t, x)}{h(t) + h^\Delta(t)\mu(t)} h^\Delta(t). \end{aligned}$$

Hence *Eq.* (3.5) holds under the above assumptions with $L = 0$. Therefore all the conditions of Theorem 3.4 are satisfied and we conclude that all solutions to *Eqs.*(4.1) satisfy (3.5).

In fact, if there is a constant K such that

$$0 \leq a\mu(t) \leq K < 1 \tag{4.3}$$

for all $t \in [0, \infty)$, then $Eq.(4.2)$ holds. $\square$

Remark 4.2.

Case 1: If $\mathbb{T} = \mathbb{R}$ then $\mu(t) = 0$ and $Eq.(4.3)$ will hold for any $0 \leq K < 1$ which, in turn, will make $Eq.(4.2)$ hold and we conclude that all solutions satisfy (3.5).

Case 2: If $\mathbb{T} = \{H_n\}_0^\infty$ defined by

$$H_0 = 0,\ H_n = \sum_{r=1}^{n} 1/r,\ n \in \mathbb{N},$$

then $\mu(t) = 1/(n+1)$ and $Eq.$ (4.3) will hold when $a < 1$ which, in turn, will make $Eq.(4.2)$ hold and we conclude that all solutions satisfy (3.5).

Case 3: If $\mathbb{T} = p\mathbb{N}_0$ then $\mu(t) = p$ and $Eq.(4.3)$ will hold when $ap < 1$ which, in turn, will make $Eq.(4.2)$ hold and we conclude that all solutions satisfy (3.5).

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